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Physics in Terms of Difference Equations

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1. Introduction

The invention of quantum mechanics by Niels Bohr, and others, brought a profound change in our thinking. As a result of quantum mechanics, all dynamics is now described in terms of observables, and each observable is expressed as an operator or a matrix. Its eigenvalues denote the possible outcome of any measurements on the observable, with a probability amplitude given by the appropriate component of the state vector. These ideas were truly revolutionary at the time of their creation. By now they are accepted universally and taught to students everywhere. Mankind will be forever grateful to Niels Bohr, the originator and the developer of quantum mechanics.

Nearly thirty years ago Pauli came to Columbia University to give a seminar on his joint work with Heisenberg. Many people were in the audience, including Niels Bohr. There were a lot of questions, discussions and criticisms, some not so friendly. In the end Pauli said, in an uncharacteristic way: "Perhaps my idea is crazy." Whereupon Bohr stood up, walked a few steps towards Pauli, looked down, pointed a finger at Pauli and said, "The problem is not whether your idea is crazy, but whether it is crazy enough."

It is in this spirit that I wish to analyze further some of the basic concepts of quantum mechanics. Let me first raise two questions:

(i) In a measurement of, say, the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ is it not true that the precise values of $\boldsymbol{r}$ and t, like $\boldsymbol{E}$, can only be determined after the measurement?

(ii) Is the concept of local field theory applicable to very small distances?

1.1. Space and time as dynamical variables

To take up the first question, consider, say, an experimentalist who proposes to the CERN program committee to measure the electric field in an e^+e^- collision at LEP in 1989. The proposal is approved. The experimentalist and his several hundred collaborators then set up their gigantic instruments. Before the measurement, they have some expectation of the amplitude of the electric field $\boldsymbol{E}$ that they are going to measure. They also know the approximate location and the time that the collision will take place. The precise value of $\boldsymbol{E}$, the exact location of the reaction and the precise time of the collision can only be determined after his measurement. In other words, in terms of observation, we do treat space–time on the same footing as the electric field. Therefore, in terms of conceptual thinking, should we not also treat space–time on the same footing as the electric field? Wouldn't it be more in the spirit of Niels Bohr to regard space and time both as *dynamical variables*, the same as the electric field?

In the conventional description of local quantum field theory, we view the fields as the dynamical variables, represented by operators, but space $\boldsymbol{r}$ and time t are only parameters. Even in Einstein's general relativity, although the metric is a dynamical variable, the continuous four-dimensional space–time that the metric is embedded in is not. In this paper I shall explore the alternative view, treating space and time as dynamical variables, playing a similar role as the electromagnetic field, the gluon field, etc. As we shall see, this new description has a connection with my second question.

1.2. Locality at the Planck length

Let me present an argument to show that the concept of local field theory is possibly inapplicable to distances of the order of the Planck length l_P. In fig. 1 we consider two points A and B, separated by a spatial distance Δx and a time difference Δt with ($\hbar = c = 1$)

$$\Delta t < \Delta x < l_P \sim 10^{-33} \text{ cm},$$

in which the first inequality ensures that A and B are outside each other's light cone. Local field theory then states that these two experiments can be done independently of each other, no matter how close the points. Yet, just based on the uncertainty principle, we expect a fluctuation of energy ΔE caused by these two measurements,

$$\Delta E \sim \frac{1}{\Delta t} > \frac{1}{l_P}.$$

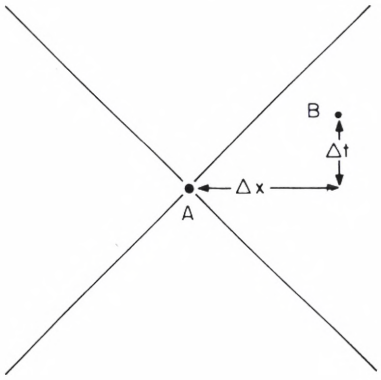

Fig. 1. Space–time points A and B outside each other's light cone.

The gravitational field associated with such a ΔE is very strong at small distances. Indeed, its Schwarzschild radius (i.e. black hole radius) is

$$R \sim G\Delta E > \frac{G}{l_{\mathrm{P}}},$$

where $G = l_{\mathrm{P}}^2$ is Newton's gravitational constant. Thus we find

$$R > \Delta x!$$

It seems quite unreasonable that these two measurements A and B could be viewed as independent. Consequently, the concept of locality very likely breaks down at such distances. Likewise, the usual linear superposition principle of quantum mechanics may also be questioned. If locality is not satisfied at the Planck length, then the correct physical theory must be non-local in character. The fact that the Planck length is small is beside the point. We would like this non-local fundamental theory to retain all the good features of the usual continuum theory: Lorentz invariance, Poincaré invariance, non-Abelian gauge symmetries, unitarity and the general coordinate invariance of general relativity. In addition it should not contain divergence difficulties, so that quantization of gravity can be carried out.

There are perhaps two different directions one may follow: one is to add degrees of freedom to the usual local field theory. The other is to subtract degrees of freedom. The former is followed by those working on string theories, and the latter will be the subject of this chapter.

2. *Time as a dynamical variable*

In this new theory I shall treat time as a dynamical variable [1]. This will lead to a dynamics which is formulated in terms of difference equations, instead of the usual differential equations. We will first review briefly the classical theory of this new mechanics, called discrete mechanics, and then go over to the quantum theory.

2.1. Classical mechanics

Take the simplest example of a non-dimensional nonrelativistic particle of unit mass moving in a potential $V(x)$. In the usual continuum mechanics the action is

$$A(x(t)) = \int_0^T \left[\tfrac{1}{2}\dot{x}^2 - V(x)\right] \mathrm{d}t, \tag{2.1}$$

where $x(t)$ can be any smooth function of time t. Keeping the initial and final positions fixed, say x_0 and x_f, at $t = 0$ and T, we determine the orbit of the particle by the stationary condition

$$\frac{\delta A}{\delta(x(t))} = 0, \tag{2.2}$$

which leads to Newton's equation

$$\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{2.3}$$

In the above, x is the dynamical variable and t is merely a parameter. Next, we shall see how this customary approach may be modified in the discrete version.

Let the initial and final positions of the particle be the same:

$$x_0 \quad \text{at } t = 0 \quad \text{and} \quad x_f \quad \text{at } t = T. \tag{2.4}$$

In the discrete mechanics we restrict the usual smooth path $x(t)$ to a "discrete path" $x_D(t)$, which is continuous but piecewise linear, characterized by N vertices (as shown in fig. 2). In fig. 2a we have the usual smooth path $x(t)$ of a nonrelativistic particle in classical mechanics. Moving along $x(t)$ from $t = 0$ to $T > 0$, the time t increases monotonically; this property is retained under the constraint restricting

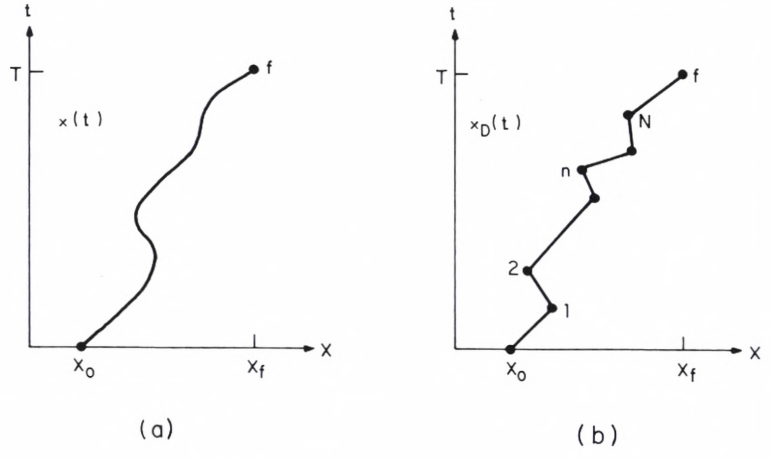

Fig. 2. Usual smooth path (a) and discrete path (b).

$x(t)$ to $x_D(t)$. Thus, as in fig. 2b, we may label the N vertices of $x_D(t)$ consecutively as $n = 1, 2, \ldots, N$, each of which carries a space–time position x_n and t_n with

$$0 < t_1 < t_2 < t_3 < \cdots < t_N < T. \tag{2.5}$$

The nearest-neighboring vertices are linked by straight lines, forming the discrete path $x_D(t)$, which also appears as a one-dimensional lattice with n as lattice sites. In fig. 2b, a variation of the space–time positions of these vertices changes the discrete path $x_D(t)$. However, a mere exchange of any two vertices clearly defines the same $x_D(t)$. This is because only the discrete path with unlabeled vertices has a physical meaning. There is no "individual" identity of any of the vertices. Thus, the time-ordered sequence (2.5) is not an additional restriction, but one that arises naturally when we pass from the usual nonrelativistic path $x(t)$ to the discrete $x_D(t)$.

In the following, we shall keep the site-density

$$\frac{N}{T} \equiv \frac{1}{l} \tag{2.6}$$

fixed, and regard l as a fundamental constant of the theory. The action integral (2.1) evaluated on such a discrete path $x_D(t)$ is

$$A_D = A(x_D(t)) = \sum_n \left[\frac{1}{2} \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} - (t_n - t_{n-1})\, \bar{V}(n) \right], \tag{2.7}$$

where

$$\bar{V}(n) = \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} V(x)\, \mathrm{d}x \tag{2.8}$$

is the average of $V(x)$ along the straight line between x_{n-1} and x_n.

Because the path $x_D(t)$ is completely specified by its vertices $n(x_n, t_n)$, a variation in $x_D(t)$ is equivalent to a variation in all the positions of its vertices

$$\mathrm{d}[x_D(t)] = \prod_n [\mathrm{d}x_n][\mathrm{d}t_n]. \tag{2.9}$$

Correspondingly, the dynamical eq. (2.2) becomes the difference equations

$$\frac{\partial A_D}{\partial x_n} = 0, \tag{2.10a}$$

and

$$\frac{\partial A_D}{\partial t_n} = 0. \tag{2.10b}$$

We see that in this new mechanics the roles of x_n and t_n are quite similar. Both appear as *dynamical variables*. For each x_n or t_n we have one difference equation, (2.10a) or (2.10b). The former gives Newton's law on the lattice and the latter gives the conservation of energy

$$E_n \equiv \frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 + \overline{V}(n) = E_{n+1}. \tag{2.11}$$

In the usual continuum mechanics, conservation of energy is a consequence of Newton's equation. Here, the two eqs. (2.10a) and (2.10b) are independent. Altogether there are $2N$ such equations, matching in number the $2N$ unknowns x_n and t_n in the problem. Because the action A_D is stationary under a variation in x_n and in t_n for all n, the discrete theory retains the translational invariance of both space and time, and that leaves the conservation laws of energy and momentum intact. *

For a free particle $V(x) = 0$, eqs. (2.10a) and (2.10b) become degenerate; both give

$$v_n = \frac{x_n - x_{n-1}}{t_n - t_{n-1}} = \text{constant}.$$

The corresponding trajectory is a straight line, the same as the continuum case.

When $V(x) = gx$ with g a constant, the solution of eqs. (2.10a) and (2.10b) can be readily found. We find in this case the spacing between successive t_n to be independent of n:

$$t_n - t_{n-1} = \epsilon = \text{constant}.$$

Correspondingly, $t_n = t_0 + n\epsilon$ and

$$x_n = x_0 + nv_1\epsilon - \tfrac{1}{2}n(n-1)g\epsilon^2,$$

where v_1 is the initial velocity $(x_1 - x_0)/(t_1 - t_0)$.

When $l \to 0$, the site density $\to \infty$ and the discrete path $x_D(t)$ can assume the form of any smooth path $x(t)$; consequently the discrete mechanics approaches the usual continuum mechanics. Introduce

$$\tau \equiv nl,$$

which varies from 0 to T, as n runs from 0 to N. Consider the quantities

$$x_n = x(\tau) \quad \text{and} \quad t_n = t(\tau).$$

From eqs. (2.10a) and (2.10b), it can be shown that in the limit $l \to 0$, but keeping T

* Here, conservation of momentum means that the change of particle momentum is equal to the "impulse" generated by the potential.

fixed (hence, $N \to \infty$),

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{\mathrm{d}V}{\mathrm{d}x}, \tag{2.12a}$$

and

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^3 \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 = \text{constant}; \tag{2.12b}$$

the former is Newton's equation, and the latter gives the asymptotic distribution of t_n versus n. The constant in eq. (2.12b) is determined by the boundary condition (2.4), so that when τ varies from 0 to T, t also changes from 0 to T. In the usual continuum mechanics, only eq. (2.12a) is retained. Therefore, even in this limit, the discrete mechanics contains more information than the usual continuum mechanics. From eq. (2.12b), we see that, except for $V(x) = gx$, the spacing $t_n - t_{n-1}$ is not a constant.

It is of interest to examine the distribution $t(\tau)$ near the point $V'(x) \equiv \mathrm{d}V/\mathrm{d}x = 0$, which occurs at, say, $x = \bar{x}$. Let the particle trajectory in the continuum limit be $x = x(t)$. When $x = \bar{x}$, we have $V'(\bar{x}) = 0$ and, for the solution under consideration, $t = \bar{t}$ so that $\bar{x} = x(\bar{t})$. In the neighborhood x near $\bar{x}$, we may write, with $V''(x) \equiv \mathrm{d}^2 V/\mathrm{d}x^2$ and $\dot{x} \equiv \mathrm{dx}/\mathrm{dt}$,

$$\begin{aligned} V'(x) &\simeq (x - \bar{x})\, V''(\bar{x}) \\ &= (t - \bar{t})\dot{x}(\bar{t})\, V''(\bar{x}). \end{aligned}$$

Substituting this expression into eq. (2.12b) we find

$$(t - \bar{t}) \propto (\tau - \bar{\tau})^{3/5}.$$

Hence, as $\tau \to \bar{\tau}$ (correspondingly $n \to \bar{\tau}/l$), although $\mathrm{d}t/\mathrm{d}\tau \to \infty$ one sees that $t \to \bar{t}$ and remains finite. Information such as this is lost if one concentrates only on Newton's equation (2.12a).

In the following, we are interested in $l \neq 0$, in which case the discrete mechanics is fundamentally different from the continuum theory.

2.2. Nonrelativistic quantum mechanics

When we go over from classical to quantum mechanics, in the usual continuum theory the particle can take on any smooth path $x(t)$; each path carries an amplitude $\mathrm{e}^{\mathrm{i}A}$ where $A = A(x(t))$ is the same action integral (2.1). In Feynman's path integration formalism, the matrix element of $\mathrm{e}^{-\mathrm{i}HT}$ in the usual continuum quantum mechanics is given by

$$\langle x_{\mathrm{f}} | \mathrm{e}^{-\mathrm{i}HT} | x_0 \rangle = \int \mathrm{e}^{\mathrm{i}A(x(t))}\, \mathrm{d}[x(t)], \tag{2.13a}$$

in which all paths $x(t)$ have the same end-points (2.4) and

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x). \tag{2.13b}$$

Sometimes it is more convenient to consider the analytic continuation of T to $-\mathrm{i}T$. The operator $\mathrm{e}^{-\mathrm{i}HT}$ becomes then e^{-HT}, and its matrix element is given by

$$\langle x_\mathrm{f} | \mathrm{e}^{-HT} | x_0 \rangle = \int \mathrm{e}^{-\mathscr{A}(x(t))}\, \mathrm{d}[x(t)], \tag{2.14}$$

where

$$\mathscr{A}(x(t)) = \int_0^T \left[\tfrac{1}{2}\dot{x}^2 + \mathrm{V}(x)\right] \mathrm{d}t. \tag{2.15}$$

In the corresponding discrete theory, we again restrict the particle to move only along the discrete path $x_\mathrm{D}(t)$. By using eqs. (2.7) and (2.9), we see that the right-hand side of eq. (2.13a) becomes

$$\int e^{\mathrm{i}A_\mathrm{D}} \prod_n [\mathrm{d}x_n][\mathrm{d}t_n]. \tag{2.16}$$

Likewise, eqs. (2.14) and (2.15) become

$$\langle x_\mathrm{f} | G_N(T) | x_0 \rangle \equiv \int \mathrm{e}^{-\mathscr{A}_\mathrm{D}} \prod_{n=1}^{N} [\mathrm{d}x_n][\mathrm{d}t_n], \tag{2.17}$$

where

$$\mathscr{A}_\mathrm{D} = \mathscr{A}(x_\mathrm{D}(t)). \tag{2.18}$$

When the vertices $n = 1, 2, \ldots,$ are arranged in a time-ordered sequence (2.5), by using eqs. (2.15) and (2.18) we see that the discrete action $\mathscr{A}_\mathrm{D}$ is given by

$$\mathscr{A}_\mathrm{D} = \sum_{n=1}^{N+1} \left[\frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})} + (t_n - t_{n-1})\, \bar{V}(n) \right], \tag{2.19}$$

with $x_{N+1} = x_\mathrm{f}$ and $t_{N+1} = T$, as shown in figure 2b.

In the integration over $\Pi_n[\mathrm{d}t_n]$, whenever t_i appears larger than, say, t_{i+1}, we should re-link the vertices so that the newly linked ones are in a time-ordered sequence. Alternatively, we may re-label them so that eq. (2.5) remains valid; such a relabeling of vertices clearly does not change the discrete path $x_\mathrm{D}(t)$. (As explained before, this follows from the usual nonrelativistic continuum mechanics in which the path $x(t)$ is a single-valued function of t.)

In the quantum version of the discrete mechanics it is more convenient to regard the constraint (2.6) as a condition on the *average* site-density. This can be most easily arranged by considering an ensemble sum over N:

$$\mathscr{G}(T, l) \equiv \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{1}{l}\right)^N G_N(T), \tag{2.20}$$

where $G_N(T)$ refers to the matrix defined by eq. (2.17). One may readily verify that this Green function satisfies

$$\frac{\partial}{\partial(1/l)} \mathscr{G}(T, l) = \int_0^T \mathscr{G}(\tau, l)\, \mathscr{G}(T-\tau, l)\, \mathrm{d}\tau, \tag{2.21}$$

from which it follows that for large T and neglecting $\mathrm{e}^{-T/l}$ the operator $\mathscr{G}(T, l)$ becomes

$$\mathscr{G}(T, l) \sim \mathrm{e}^{-\mathscr{H}T}, \tag{2.22}$$

where $\mathscr{H}$ is Hermitian. When $l \to 0$, $\mathscr{H}$ reduces to the continuum Hamiltonian H, given by eq. (2.13b). The analytic continuation of $\mathscr{G}(T, l)$ from T to $\mathrm{i}T$ leads at large T to the unitary operator $\mathrm{e}^{-\mathrm{i}\mathscr{H}T}$, which is the S-matrix of the theory. Therefore, the unitarity of the S-matrix is maintained in the new mechanics [2], at least when $\mathscr{H}$ is O(1).

3. *Relativistic quantum field theory*

As an example, let $\phi(x)$ be a scalar field in the usual continuum theory with x denoting the space–time coordinates. In the path integration formulation the operator e^{-HT} is given by, similar to eq. (2.14),

$$\mathrm{e}^{-HT} = \int \mathrm{e}^{-\mathscr{A}} [\mathrm{d}\,\phi(x)], \tag{3.1}$$

where H is the Hamiltonian operator, $\mathscr{A}$ the usual continuum action in the Euclidean space and T the total "time" interval. (Here, as in eqs. (2.14) and (2.15), "time" refers to the Euclidean time.) Because in the usual continuum theory the space–time coordinates x are parameters and only $\phi(x)$ are dynamical variables, the functional integration in eq. (3.1) is over $[\mathrm{d}\,\phi(x)]$, not over $[\mathrm{d}x]$.

In the discrete version, we impose a constraint on the (average) number N of experiments that can be performed within any given space–time volume Ω, with $N/\Omega \equiv l^{-4}$ = fundamental constant. Each measurement determines the field $\phi(i)$ as well as the space–time position $x(i)$ with $i = 1, 2, \cdots, N$. The i will be referred to as lattice sites, as illustrated by fig. 3a.

As we shall see, the Green function (3.1) will be replaced by

$$\int \mathrm{e}^{-\mathscr{A}_{\mathrm{D}}} [\mathrm{d}\,x(i)]\,[\mathrm{d}\,\phi(i)]. \tag{3.2}$$

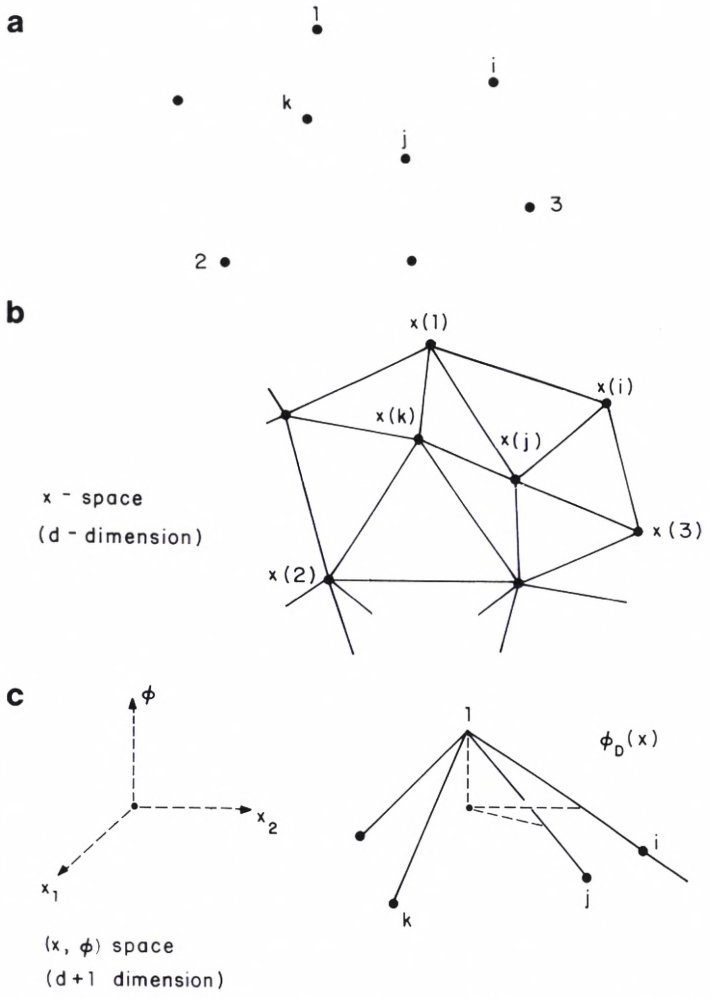

Fig. 3. (a) Lattice sites in space–time. (b) Lattice sites coupled to neighboring sites. (c) The "discrete" function $\phi_D(x)$ in $x-\phi$ space.

Because $\phi(i)$ and $x(i)$ are all dynamical variables, in the discrete theory we integrate over [d $\phi(i)$] as well as over [d $x(i)$]. The latter integration makes it obvious that rotational and translational symmetries can be maintained in the discrete theory.

To simulate the local character of the usual continuum theory, each site in the discrete theory is coupled only to its neighboring sites, as illustrated in fig. 3b. The whole volume is then divided into triangles if the dimension of $x(i)$ is $d=2$, tetrahedra if $d=3$, four-simplices when $d=4$, etc. An example of such a simplicial lattice when $d=2$ appears in fig. 3b.

We give the algorithm [3] of linking an arbitrary distribution of sites into a simplicial lattice for any dimension d. Select any group of $d+1$ sites; consider the

hypersphere (in the d-dimensional Euclidean space) whose surface passes through these $d+1$ sites. If the interior of the sphere is empty of sites, link these sites to form a d-simplex; otherwise, do nothing. Proceed to another group of $d+1$ sites, and repeat the same steps. The d-simplices thus formed never intersect each other, and the sum total of their volumes fills the entire space.

Each site i carries, in addition to its space–time coordinates $x(i)$, also a $\phi(i)$. Viewed in the x–ϕ space, the lattice forms a d-dimensional surface represented by $\phi_{\mathrm{D}}(x)$, called the "discrete" function; it is continuous but piece-wise flat within each d-simplex as illustrated in fig. 3c.

The discrete action $\mathscr{A}_{\mathrm{D}}$ in eq. (3.2) can be readily evaluated by using the usual continuum action $\mathscr{A}(\phi(x))$, but restricting $\phi(x)$ to the discrete function:

$$\mathscr{A}_{\mathrm{D}} \equiv \mathscr{A}(\phi_{\mathrm{D}}(x)). \tag{3.3}$$

For example, if

$$\mathscr{A}(\phi(x)) = \int \left[\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right] \mathrm{d}x, \tag{3.4}$$

where $\mathrm{d}x$ is the d-dimensional volume element in the x-space, then setting $\phi(x)$ to be the discrete function $\phi_{\mathrm{D}}(x)$, we find

$$\mathscr{A}_{\mathrm{D}} = \mathscr{A}(\phi_{\mathrm{D}}(x)) = \tfrac{1}{2}\sum_{l_{ij}} \lambda_{ij}\left([\phi(i) - \phi(j)]^2\right) + \sum_i \omega_i V(\phi(i)), \tag{3.5}$$

where the first sum is over all links l_{ij} and the second over all sites i, ω_i is the volume of the Voronoi cell that is dual to the site i, and [4]

$$\lambda_{ij} = -\frac{1}{d^2}\sum \frac{1}{V(ij)} \boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j), \tag{3.6}$$

in which the sum extends over all d-simplices $V(ij)$ that share the link l_{ij}. In $V(ij)$, each vertex, say k, faces a $(d-1)$-dimensional simplex $\tau(k)$. In eq. (3.6), $V(ij)$ denotes also the volume of the d-simplex and $\boldsymbol{\tau}(i)$ is the outward normal vector of $\tau(i)$ times its $(d-1)$-dimensional volume, as illustrated in figure 4. As in the previous section, mathematically the discrete theory can be regarded as a special case of the usual continuum theory: one in which $\phi(x)$ is restricted to those continuous but piece-wise flat functions $\phi_{\mathrm{D}}(x)$ with a fixed average density of vertices (i.e. lattice sites). Because the site density is an invariant, rotational and translational invariances can both be preserved in the discrete theory.

Since the discrete surface, described by $\phi_{\mathrm{D}}(x)$, is characterized by the positions $\phi(i)$ and $x(i)$ of its vertices, a variation over the functional space $\phi(x)$ in the usual continuum theory becomes

$$[\mathrm{d}\,\phi_{\mathrm{D}}(x)] = \prod_i [\mathrm{d}\,\phi(i)][\mathrm{d}\,x(i)]. \tag{3.7}$$

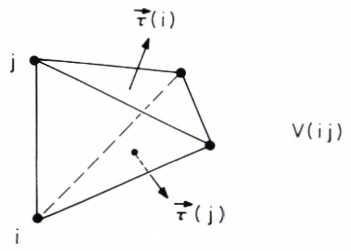

Fig. 4. Simplex $V(ij)$ and the associated outward normals $\tau(i)$.

Correspondingly, eq. (3.1) becomes eq. (3.2). As $x(i)$ changes, the linking algorithm keeps track of how these vertices should be linked, so that the discrete action $\mathscr{A}_D$ is extensive; i.e. $\mathscr{A}_D$ is proportional to the overall space–time volume Ω when Ω is large. Thereby, the unitarity of the S-matrix can be established, as before.

In the usual continuum theory, the equation of motion is given by the partial differential equation

$$\frac{\delta \mathscr{A}(\phi(x))}{\delta \phi(x)} = 0. \tag{3.8}$$

Here in the discrete version it is replaced by the difference equations

$$\frac{\partial \mathscr{A}_D}{\partial \phi(i)} = 0 \quad \text{and} \quad \frac{\partial \mathscr{A}_D}{\partial x(i)} = 0; \tag{3.9}$$

the former is the field equation on the lattice and the latter expresses the conservation law of the energy–momentum tensor.

In the integrand of eq. (3.2), the locations of x can be arbitrary. Hence, the discrete action $\mathscr{A}_D$ is identical to that of a random lattice [5].

4. Gauge theory

We review briefly the random lattice results on Abelian (QED) and non-Abelian (QCD) gauge theories.

The lattice gauge theory was introduced by K. Wilson. In the strong-coupling limit (square of coupling constant $g^2 \to \infty$), any lattice gauge theory gives confinement. This holds for both QED and QCD, and for arbitrary space dimension d. The realistic case corresponds, however, to the weak coupling. Thus, a key question is whether the transition from strong to weak coupling is smooth or not. If smooth, then the confinement property of the strong coupling can be carried over to weak coupling, otherwise not. When $\beta = 1/g^2$ changes from 0 (strong coupling) to ∞ (weak coupling), we would like the transition to be smooth for the non-Abelian case, but not smooth for the Abelian, so that the confinement holds for QCD, but not for QED. In a hypercubic lattice, there appears to be a phase transition in β for the

U(1) gauge, consistent with the fact that QED is not confined. However, numerical results in SU(2) and SU(3) indicate that the transition from $\beta = 0$ to $\beta = \infty$ is also far from smooth for a hypercubic lattice. While there is probably no bona fide phase transition in the non-Abelian case, the change from cubic ($g^2 = \infty$) symmetry to spherical ($g^2 = 0$) symmetry is sufficiently hazardous that it is difficult to infer, from the strong-coupling result, that confinement would remain valid in the weak coupling.

On the other hand, for the random lattice, its strong-coupling limit behaves like a relativistic string theory, with full rotational symmetry: the string thickness t is related to the string tension T by

$$t^2 = \frac{1}{2\pi T} \ln a, \tag{4.1}$$

where a is the area enclosed by the string. Furthermore, the mass of the glueball m_J

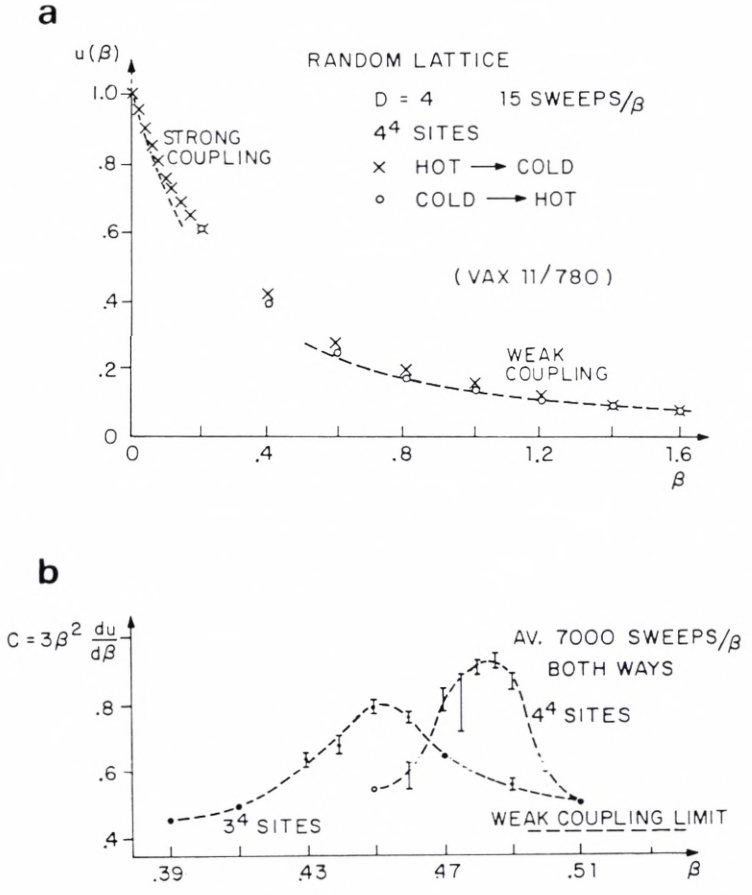

Fig. 5. (a) Average plaquette energy u and (b) specific heat C in $U(1)$ theory, as functions of $\beta = g^{-2}$.

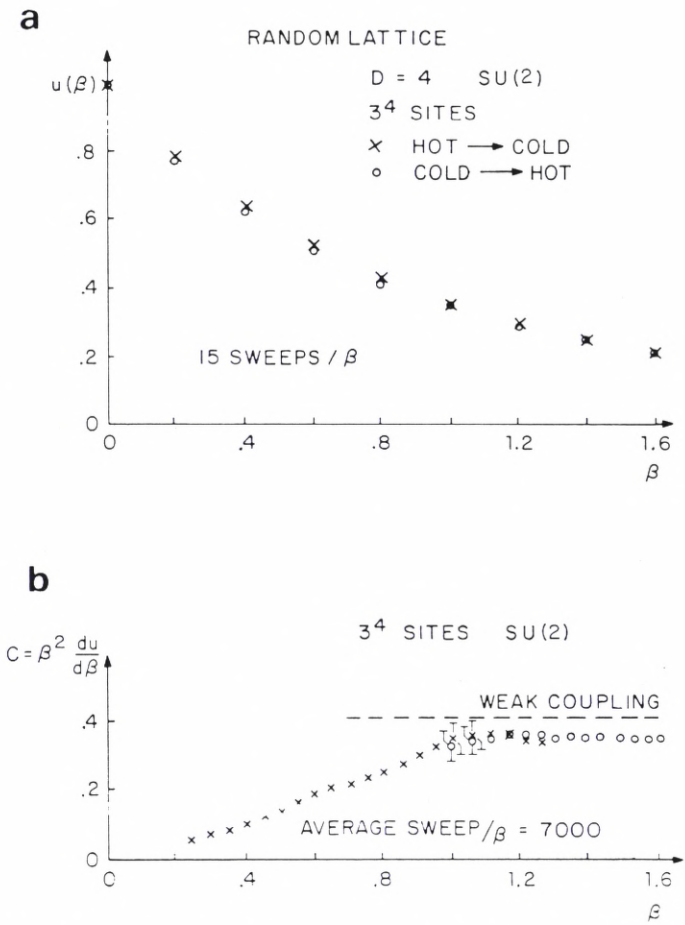

Fig. 6. (a) Average plaquette energy u and (b) specific heat C in SU(2) theory, as functions of $\beta = g^{-2}$.

for large angular momentum J varies as

$$m_J \propto \sqrt{J}\,, \tag{4.2}$$

exhibiting the typical Regge behavior of a rotating relativistic string. Both eqs. (4.1) and (4.2) are valid in the strong-coupling limit.

Numerical programs for a random lattice gauge theory were set up by Friedberg and Ren at Columbia; the computations were carried out by Ren [6]. In fig. 5 we give the average plaquette energy u and specific heat C vs. $\beta = 1/g^2$ for the U(1) theory.

The corresponding plots for an SU(2) theory are given in fig. 6. We see that the specific heat has a peak in the U(1) theory, but not in the SU(2) theory. For U(1), the peak becomes steeper when the number of lattice sites increases, suggesting that

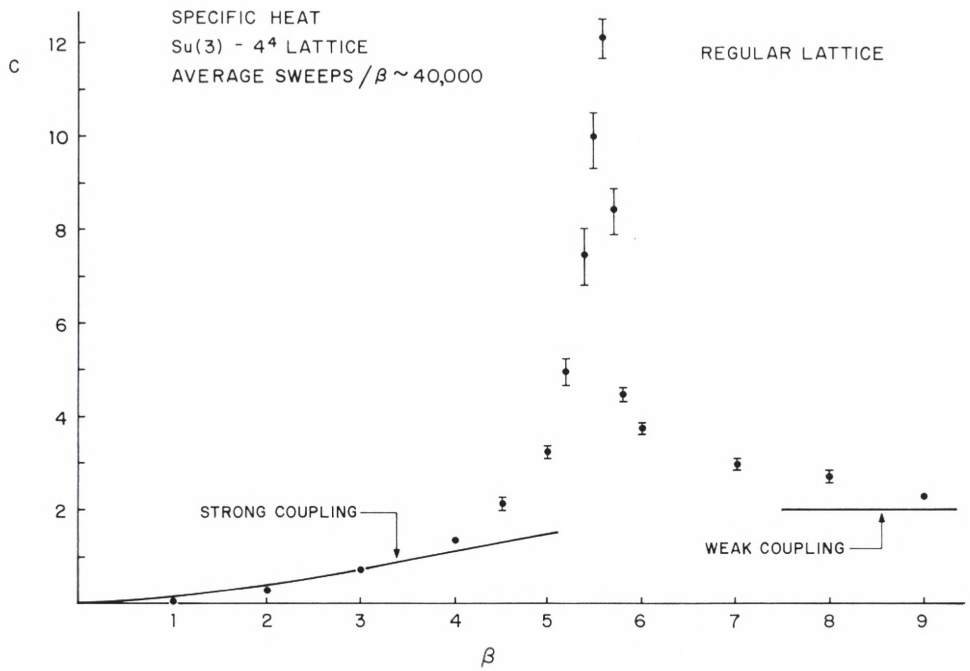

Fig. 7. Specific heat for SU(3) gauge theory on a regular lattice.

there is a phase transition. On the other hand, the specific heat curve for SU(2) has no peak, indicating that the passage from strong to weak coupling is a smooth one. Consequently, while both theories are confined in the strong coupling limit, the weak coupling limit is consistent with deconfinement in the U(1) theory (QED), but with confinement in a non-Abelian gauge theory (QCD).

In contrast, we give in fig. 7 the numerical calculation by N.H. Christ and A.E. Terrano [7] for the SU(3) gauge theory on a regular lattice. As we can see, there is a sharp peak in the specific heat, suggesting that the transition from strong to weak coupling in a regular lattice is by no means smooth, unlike that in a random lattice.

5. *Lattice gravity*

The usual Einstein action in general relativity is

$$A(S) = \int_S \sqrt{|g|}\,\mathscr{R}\,\mathrm{d}x, \tag{5.1}$$

where S is a d-dimensional smooth continuous surface, $|g|$ is the absolute value of the determinant of the matrix of the metric tensor $g_{\mu\nu}$ on S, $\mathscr{R}$ is the scalar curvature and $\mathrm{d}x$ is the d-dimensional volume element in the space–time coordinate x.

For lattice gravity, we consider first a (random) lattice $\mathring{L}$ in a flat d-dimensional Euclidean space R_d. Label each site by $i = 1, 2, \ldots$. For every linked pair of sites i and j there is a link-length $\mathring{l}_{ij}$.

Consider now an arbitrary variation

$$\mathring{l}_{ij} \to l_{ij}. \tag{5.2}$$

Correspondingly, each d-simplex, say $\mathring{\tau}$ in $\mathring{L}$, becomes a new d-simplex τ with the same vertices, but different link-lengths. These new link-lengths l_{ij} are assumed to satisfy all simplicial inequalities, so that each d-simplex τ, by itself, can still be realized in a flat d-dimensional space R_d. In general the entire new lattice cannot fit into R_d. This then defines [8] a d-dimensional non-flat lattice surface L.

Sometimes, it is convenient to embed L in a flat space R_N. This is possible if

$$N = d + n$$

is sufficiently large; in that case

$$l_{ij}^2 = [\boldsymbol{r}(i) - \boldsymbol{r}(j)]^2 \quad \text{in } R_N, \tag{5.3}$$

with $\boldsymbol{r}(i)$ the Cartesian N-dimensional position vector of the ith site in R_N. Since, as we shall see, we shall deal only with the intrinsic geometric properties of the lattice surface, this embedding is merely a convenience.

Next we wish to evaluate Einstein's action (5.1) when S is restricted to the lattice surface L. At first sight, it might appear difficult because the metric g_{ij} would change discontinuously from simplex to simplex, the Christoffel symbol would then acquire δ-functions, and the scalar curvature δ'-functions. Since Einstein's action is nonlinear in g_{ij}, one might expect the resulting expression to be totally unmanageable. It turns out that this is not the case.

It can be shown that the Einstein formula (5.1) evaluated on any d-dimensional lattice space L gives the discrete action [9,10]

$$A(L) \equiv \int_L \sqrt{|g|}\, \mathscr{R}\, \mathrm{d}x \tag{5.4}$$

$$= 2 \sum_s s \epsilon_s, \tag{5.5}$$

where $\mathrm{d}x$ is the d-dimensional volume element, s is the volume of the $(d-2)$ simplex, ϵ_s is Regge's deficit angle around s and the sum extends over all s in the lattice. (See ref. [11] for the definition of ϵ_s.) The right-hand side of eq. (5.5) is *precisely* the formula of Regge calculus [11].

In Regge's original approach, he considered the discrete action as an approximation to Einstein's continuum action. Here we are reversing the role and regarding the discrete action $A(L)$ as more fundamental. It is therefore satisfying to realize that Regge's action is identical to Einstein's action, but evaluated on L.

The quintessance of Einstein's theory of general relativity lies in its invariance under a general coordinate transformation

$$x \to x' \tag{5.6}$$

that leaves ds^2 unchanged. Since the action for the lattice space L is the discrete action

$$A_D \equiv A(L) = \int_L \sqrt{|g|}\,\mathscr{R}\,dx = 2\sum_s s\epsilon_s, \tag{5.7}$$

the discrete theory clearly remains invariant under the coordinate transformation (5.6). Thus, the entire apparatus of coordinate invariance in the usual continuum theory automatically applies to the lattice theory as well. In addition, as we shall see, the lattice theory enjoys still another total new class of symmetries which does not exist in the usual continuum theory. Aesthetically, this adds greatly to the appeal of lattice gravity. For physical applications, when the link-length l is small, our general formula (5.7) ensures that all known tests of general relativity are automatically satisfied. Furthermore, by keeping l nonzero, we see that the lattice action A_D per volume possesses only a finite degree of freedom. The normal difficulty of ultra-violet divergence that one encounters in quantum gravity disappears in the lattice theory. All these suggest that the lattice theory with a nonzero l may be more fundamental. The usual continuum theory is quite possibly only an approximation.

To amplify the aforementioned symmetry properties, let us consider any lattice L. From eq. (5.7), we see that the discrete action A_D, through its right-hand side, is a function of the link-lengths l_{ij},

$$A_D = A_D(l_{ij}). \tag{5.8}$$

We may also characterize the lattice by other means of parametrization. We assume all the lattice sites i to lie on a d-dimensional smooth enveloping surface S, with $z_\mu(i)$ as the coordinates of the site i on S, where $\mu = 1, 2, \ldots, d$. Embed both S and L in a flat space R_N, which is always possible provided that N is sufficiently large. Because of eq. (5.3), l_{ij} can also be determined by giving S and $z_\mu(i)$. Hence we can also express A_D as a function of the enveloping surface S and the site positions on S,

$$A_D = A_D\big(S,\ z_\mu(i)\big). \tag{5.9}$$

Thus, we can have new symmetry transformations,

(i) fix $z_\mu(i)$, but vary $S \to S'$,

(ii) fix S, but vary $z_\mu(i) \to z'_\mu(i)$.

These symmetries are exact if l_{ij} are unchanged; they can be approximate even if l_{ij} does change, provided that, e.g. the link-lengths are sufficiently small and $\sqrt{|g|}\,\mathscr{R}\,dx$ remains the same on the enveloping surface, in which case $A_D \simeq A(S)$ of eq. (5.1).

In the usual continuum theory, the physical space–time points and the underlying four-dimensional manifold are the same. Here, they are distinct; the former is related to measurements, while the latter is purely a mathematical artifice (like the choice of gauge in the usual continuum theory of a spin 1, or 2, field).

6. Concluding remarks

For more than three centuries we have been influenced by the precept that fundamental laws of physics should be expressed in terms of differential equations. Difference equations are always regarded as approximations. Here, we try to explore the opposite: Difference equations are more fundamental, and differential equations are regarded as approximations.

As we have shown, such a difference equation formulation leads to the discrete mechanics which can also be viewed as the mathematical limit of the usual continuum mechanics, but with a fixed density of lattice sites. Because this is an invariant constraint, the discrete theory shares the same symmetries of the usual continuum theory. In this way, we have succeeded in the creation of theories with finite degrees of freedom, but which retain the good properties of the usual continuum theory. We suggest that this discrete formulation might be more fundamental.

In this new mechanics, space and time are treated as dynamical variables, on the same footing as electromagnetic fields, gluon fields, etc. In this sense, I hope the thinking of Niels Bohr can be extended further to bring our theoretical concepts even closer to actual experimental observation.

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Discussion, session chairman A. Salam

Regge: You are right in saying that the approximation used in the 1961 paper was designed to get away from coordinate systems. However, I would be surprised if this

system represents reality since it does not cure the diseases of the conventional formulation of quantum gravity. For example, there still is the problem that the action does not have a lower bound. So further work appears to be necessary.

Lee: Well, I hope that this is the beginning of the work and not the end.

Rubia: To those of us in experimental science, can you say what we should be looking for?

Lee: What I could tell you at this time is not yet mature. If we introduce a fundamental length which does not violate Poincaré invariance, it is reasonable that it should be somewhere between 10^{-16} cm and the Planck length, 10^{-33} cm. The worst situation we can imagine is that it is near the latter. Now, consider a system with a strong gravitational field, namely a black hole. The size of the black hole will decrease due to Hawking radiation. In conventional theory, it will ultimately vanish, but not in the theory which I have described. To answer your question, we can ask what the probability is of a proton of 10^{-13} cm dimension to shrink suddenly to, say, 10^{-33} cm and to become a tiny (lattice) black hole of a smaller mass plus soft radiation. A crude estimate gives the probability as the ratio of these two volumes, about $(10^{-33}/10^{-13})^3 = 10^{-60}$. The natural time scale is 10^{-23} s, determined by the larger dimension 10^{-13} cm. This gives a proton lifetime of about 10^{30} years, which is experimentally accessible.

Casimir: I remember vaguely that Heisenberg played with the idea of a discrete space–time around 1930. Perhaps Weizsäcker knows more about this.

Weizsäcker: It is true that in the thirties Heisenberg thought about the idea of a fundamental length, but he did not develop it, and his last theory did not contain it. But let me ask whether there is any connection between your theory and the ideas of David Finkelstein, published in 1968.

Lee: Finkelstein did have similar thoughts, but his work did not offer a concrete proposition. I should say that what I have discussed is not meant to be the ultimate theory (as I said in response to Professor Regge's remark), but only a new way of thinking and a method of getting rid of divergences while maintaining continuum symmetries.

Kohn: A continuum trajectory, in your first example, can be approximated by discrete points in many different ways. So isn't there an ill condition in your equations, leading to indeterminacy?

Lee: No.